Neutrally buoyant particle in the boundary layer at a plate. II. Inertial effects

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Abstract: This study deals with the asymptotic modeling of the disturbance flow field created by a neutrally buoyant solid particle suspended in a hydrodynamic boundary layer past a plate. The range of dimensions studied is $L/\mathrm{Re}_i^{5/4} < R_p < L/\mathrm{Re}_i^{1/2}$ (Re_i being the total Reynolds number of the background flow stream, L is the length of the plate). The concept of local particle's Reynolds number Re_p is introduced. The obtained asymptotic equations for the disturbance field are classified in three basic groups: viscous perturbation ($\mathrm{Re}_p < 1/\mathrm{Re}_i^{1/2}$); intermediate case ($\mathrm{Re}_p \sim 1/\mathrm{Re}_i^{1/2}$) and inertial perturbation ($\mathrm{Re}_p > 1/\mathrm{Re}_i^{1/2}$).

Key words: Neutral buoyancy – boundary layer – inertial effects

1. Introduction

The interaction of suspended particles with external flow fields is of great theoretical and practical interest in suspension rheology [1–4]. The complex form of the mathematical formulation of the problem however, has turned the solution of this task into a serious problem. The basic characteristics of the phenomenon is the coupling of at least two flow fields: the background stream and the flow created by the motion of the particles in stagnant fluid. The simplest case is that of neutrally-buoyant solids. The interaction with the external stream is then due only to the finite dimensions of the particles. The disturbance field is mathematically modeled by complicated differential equations even for simple Poiseulle and Couette background flows [1–4]. For complex flow streams, the situation is expected to be even more obscure.

In our previous paper [5], we have started the theoretical treatment of the disturbance flow of a solid sphere suspended in a boundary layer past a plate. The approach there was based on preliminary scaling considerations and the consequent analysis of the mathematical model. The study leads to a procedure for effective simplification of the general disturbance-field equations. A well-balanced model for the interaction was proposed, which resulted in a comparatively simple mathematical formulation of the problem. It concerned smaller particles, whose dimensions are in the range of:

$$\frac{R_p}{L} < \frac{1}{\mathrm{Re}_i^{5/4}}, \quad \mathrm{Re}_i = \frac{U_\infty L}{v}, \tag{1}$$

where R_p is the radius of the particle, L is the length of the plate, and Re_i is integral Reynold's number of the background stream. The disturbance field caused by these particles exhibits prevailing viscous character.

For larger particles, whose radii pertain to the domain

$$\frac{1}{\mathrm{Re}_i^{5/4}} < \frac{R_p}{L} < \frac{1}{\mathrm{Re}_i^{1/2}} \tag{2}$$

the scaling proposed in [5] is not adequate.

The aim of the present paper is to study the disturbance flow of particles suspended in the external boundary-layer flow and whose dimension is in the range (2). The latter is divided into three regions and the respective systems of asymptotic equations are presented.

2. Scaling considerations

A solid sphere is freely suspended in a boundary-layer flow, formed by a stream with velocity U_∞, past a plate of length L (Fig. 1). The particle is neutrally buoyant, i.e., $(\Delta\rho/\rho) < Fr^2$ ($\Delta\rho$ is the density difference of the particle and the fluid, $Fr = (U_\infty^2/Lg)^{1/2}$ is the Froude number for the external flow field). The relation (2) is valid for solid's radius R_p.

The disturbance of the basic stream flow by the solid sphere is weak. So, the total velocity and the pressure fields $(\tilde{u}, \tilde{v}, \tilde{p})$ may be presented thus:

$$\tilde{u} = \bar{u} + u' + u, \quad \tilde{v} = \bar{v} + v' + v,$$

$$\tilde{p} = \bar{p} + p' + p \,, \tag{3}$$

where the following notations are used: $(\bar{u}, \bar{v}, \bar{p})$ – for the potential flow outside the boundary-layer region; (u', v', p') – for the velocity components and the pressure inside the boundary layer. According to Blasius [6]:

$$u' = U_\infty f'(\eta), v' = \frac{1}{2}\left(\frac{\nu U_\infty}{x}\right)^{1/2}(\eta f'(\eta) - f(\eta))\,, \tag{4}$$

with ν being the kinematic viscosity of the fluid;

$$\eta = y(U_\infty/\nu x)^{1/2} \tag{5}$$

is a dimensionless variable and the function $f(\eta)$ may be approximated with a power series of η $f(\eta) = a\eta^4 + b\eta^3 + c\eta^2 + d\eta + e, a, b, c, d, e$ being constants. u, v, p stand for the disturbance field.

Upon introducing the expressions (3) in the Navier–Stokes equations of motion [7] a system of differential equations for the disturbance field is obtained:

$$(\bar{u} + u')\frac{\partial u}{\partial x} + \frac{\partial u'}{\partial x}u + v'\frac{\partial u}{\partial y} + \frac{\partial u'}{\partial y}v + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{6a}$$

$$(\bar{u} + u')\frac{\partial v}{\partial x} + \frac{\partial v'}{\partial x}u + v'\frac{\partial v}{\partial y} + \frac{\partial v'}{\partial y}v + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial x}$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{6b}$$

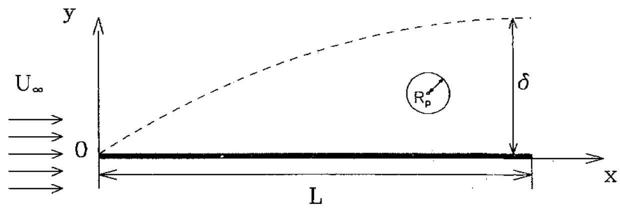

Fig. 1. Sketch of the problem. U_∞ – the background velocity; L – length of the plate; δ – boundary-layer thickness; R_p – radius of the solid sphere

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,. \tag{6c}$$

The starting point here is the choice of the characteristic parameters, which render the disturbance field dimensionless.

In the previous paper [5], the interaction was assumed to have a deformational viscous character and the disturbance velocities have been scaled with the lagging of Faxen's type:

$$u_f \approx U_\infty \varepsilon^2(1 + \Delta^2), \quad v_f \approx U_\infty \Delta \varepsilon^2(1 + \Delta^2)\,. \tag{7}$$

This case was studied in [5] and results in an asymptotic modeling of the interaction for smaller particles with dimensions in the range (1).

Any lagging with respect to the external flow, however, results in a migration in the perpendicular direction [8]. This effect is of combined inertial-viscous nature. If migration is prevailing, other characteristic quantities are obtained:

$$u_s \approx v_f \varepsilon \Delta^{1/2}, \quad v_s \approx u_f \varepsilon \Delta^{-1/2}\,. \tag{8}$$

The relation (1) is fulfilled if $u_f, v_f > u_s, v_s$. It is readily seen that always $u_f > u_s$. But it is possible to have $v_f < v_s$, that is, not to scale the transversal disturbance velocity with the viscous deformational (Faxen's) lagging, but rather with Saffman's migrational rate. This actually concerns the range of particles' dimensions Eq. (2) (Fig. 2). So, the respective disturbance quantities are rendered dimensionless via the relations:

$$u = \frac{u}{U_\infty \varepsilon^2(1 + \Delta^2)},$$

$$v = \frac{v}{U_\infty \varepsilon^3 \Delta^{-1/2}(1 + \Delta^2)}\,, \tag{9}$$

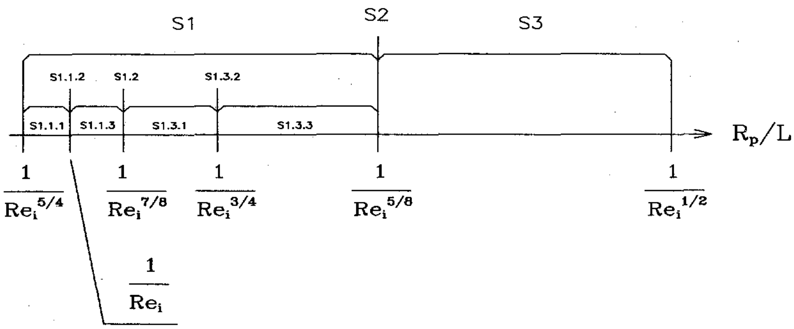

Fig. 2. Ranges of the particle dimensions. S1 – viscous interaction; S2 – intermediate case; S3 – inertial effects; $\mathrm{Re_i}$ – total Reynolds number of the background flow; L – length of the plate; R_p – radius of the solid sphere

with

$$\varepsilon = \frac{R_p}{\delta}, \quad \Delta = \frac{\delta}{L}, \quad \varepsilon, \Delta \ll 1 \,, \tag{10}$$

where ε characterizes the relative dimensions of the particle and the layer; Δ is the dimensionless thickness of the boundary layer.

We have estimated the dimensions of the disturbance field by the expressions:

$$\Delta x = \Delta x \frac{R_p}{\Delta}, \quad \Delta y = \Delta y R_p \,. \tag{11}$$

The scaling parameters in (9) and (11) are used for rendering Eqs. (6) dimensionless. The result is:

$$\mathrm{Re_p}\left\{(\bar{u} + u')\frac{\partial u}{\partial x} + \frac{\Delta^{3/2}}{\varepsilon} v' \frac{\partial u}{\partial y} + \Delta^{3/2}\frac{\partial u'}{\partial x} u + \varepsilon \frac{\partial u'}{\partial y} v + \varepsilon^2(1 + \Delta^2)\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)\right\}$$
$$= -\left((\Delta + \varepsilon\Delta + \varepsilon^2 + \varepsilon^2\Delta + \mathrm{Re_p}(1 + \varepsilon + \varepsilon^2 + \varepsilon^2\Delta^2)\right)\frac{\partial p}{\partial x} + \varepsilon^2\frac{\partial^2 u}{\partial x^2} + \Delta\frac{\partial^2 u}{\partial y^2} \tag{12a}$$

$$\mathrm{Re_p}\left\{(\bar{u} + u')\frac{\partial v}{\partial x} + \frac{\Delta^{3/2}}{\varepsilon} v' \frac{\partial v}{\partial y} + \frac{\Delta^3}{\varepsilon}\frac{\partial v'}{\partial x} u + \Delta^{3/2}\frac{\partial v'}{\partial y} v + \varepsilon^2(1 + \Delta^2)\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right)\right\}$$
$$= -\left(\Delta + \varepsilon\Delta + \varepsilon^2 + \varepsilon^2\Delta + \varepsilon\Delta^{5/2} + \mathrm{Re_p}(1 + \varepsilon^2 + \varepsilon^2\Delta^2)\right)\frac{\partial p}{\partial y} + \varepsilon^2\frac{\partial^2 v}{\partial x^2} + \Delta\frac{\partial^2 v}{\partial y^2} \tag{12b}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \,. \tag{12c}$$

The above equations contain the quantity

$$\mathrm{Re_p} = \varepsilon^2 \Delta^{-1/2} \,, \tag{13}$$

which can be interpreted as local Reynolds number for the particles in the boundary layer flow. This parameter is quite different from the respective quantity in Part I (Eq. (16) in [5]) and does not explicitly contain the integral Reynolds number ($\mathrm{Re_i}$). Three possibilities for the type of the disturbance field may appear:

case S1 – $\mathrm{Re_p} < 1$ – viscous interaction

case S2 – $\mathrm{Re_p} \approx 1$ – intermediate case

case S3 – $\mathrm{Re_p} > 1$ – inertial effects .

The Blasius solution (Eqs. (4), (5)) leads to the following expression for the boundary-layer thickness [6]:

$$\delta(x) = \mathrm{const}\sqrt{\frac{\nu x}{U_\infty}} \,. \tag{14}$$

This formula, the expression (2), and Eq. (13) give the limits of the region inside the boundary layer where the present study is adequate:

$$R_p \varepsilon \frac{\sqrt{\mathrm{Re}_i}}{\mathrm{const}} \le x \le R_p \varepsilon \frac{\mathrm{Re}_i^2}{\mathrm{const}}$$

for cases S1 and S2 , (15a)

$$R_p \varepsilon \frac{\sqrt{\mathrm{Re}_i}}{\mathrm{const}} \le x \le \frac{L}{\mathrm{const}} \mathrm{Re}_i$$

for case S3 , (15b)

where x is measured along the length of the plate. Unlike the viscous interactions in Part I (Eq. (13) in I), the realm of validity of the analysis here depends on the local Reynolds number. For small and intermediate values of Re_p, both limits explicitly contain the solid's dimensions, while at high Re_p the upper limit is determined only by the parameters of the background flow.

So, in contrast with the case of viscous-type interactions [5], the local Reynolds number here may acquire any possible value. In what follows, the above three cases are studied in detail.

3. Small local Reynolds numbers ($\mathrm{Re}_p < 1$)

There are several subcases, depending on Re_p and the other small parameters in the system. The structure of the dimensionless disturbance equations (12) implies several relations.

S1.1. $\mathrm{Re}_p < \Delta$, i.e., $\frac{1}{\mathrm{Re}_i^{5/4}} < \frac{R_p}{L} < \frac{1}{\mathrm{Re}_i^{7/8}}$

S1.2. $\mathrm{Re}_p \approx \Delta$, i.e., $\frac{1}{\mathrm{Re}_i^{5/4}} < \frac{R_p}{L} \approx \frac{1}{\mathrm{Re}_i^{7/8}}$

S1.3. $\mathrm{Re}_p > \Delta$, i.e., $\frac{1}{\mathrm{Re}_i^{7/8}} < \frac{R_p}{L} < \frac{1}{\mathrm{Re}_i^{5/8}}$.

This differentiation outlines the relative importance of the particles' dimensions and the boundary-layer scaling parameter Δ. In S1.1. the interaction is determined by the properties of the background field, while in S1.3. the sphere's radius plays the governing role (via Re_p).

S1.1.: $\mathrm{Re}_p < \Delta$, i.e., $\frac{1}{\mathrm{Re}_i^{5/4}} < \frac{R_p}{L} < \frac{1}{\mathrm{Re}_i^{7/8}}$

S1.1.1: $\varepsilon < \Delta$, i.e., $\frac{1}{\mathrm{Re}_i^{5/4}} < \frac{R_p}{L} < \frac{1}{\mathrm{Re}_i}$.

The respective dimensionless velocity components and the pressure are presented with the following power series in the small parameters ε and Δ:

$$u = u_0 + (\varepsilon^2/\Delta^{3/2})u_1 + \varepsilon u_2$$
$$v = v_0 + (\varepsilon^2/\Delta^{3/2})v_1 + \varepsilon v_2$$
$$p = p_0 + (\varepsilon^2/\Delta^{3/2})p_1 + \varepsilon p_2 \,. \quad (16)$$

Upon introducing the expansions (16) in (12), the following asymptotic expressions for the disturbance field are obtained:

S1.1.1.1 – initial approximation

$$\frac{\partial p_0}{\partial x} = \frac{\partial^2 u_0}{\partial y^2}, \quad \frac{\partial p_0}{\partial y} = \frac{\partial^2 v_0}{\partial y^2}, \quad \frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0 \,; \quad (17)$$

S1.1.1.2 – first approximation

$$(\bar{u} + u')\frac{\partial u_0}{\partial x} = -\frac{\partial p_0}{\partial x} - \frac{\partial p_1}{\partial x} + \frac{\partial^2 u_1}{\partial y^2}$$
$$(\bar{u} + u')\frac{\partial v_0}{\partial x} = -\frac{\partial p_0}{\partial y} - \frac{\partial p_1}{\partial y} + \frac{\partial^2 v_1}{\partial y^2}$$
$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0 \,; \quad (18)$$

S1.1.1.3 – second approximation

$$v'\frac{\partial u_0}{\partial y} = -\frac{\partial p_0}{\partial x} - \frac{\partial p_2}{\partial x} + \frac{\partial^2 u_2}{\partial y^2}$$
$$v'\frac{\partial v_0}{\partial y} = -\frac{\partial p_0}{\partial y} - \frac{\partial p_2}{\partial y} + \frac{\partial^2 v_2}{\partial y^2}$$
$$\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} = 0 \,. \quad (19)$$

These results are comparable with the case F.3. in [5] where the disturbance velocity is scaled with the Faxen's deformational lagging. This ensures the continuity of the behavior of the particles pertaining to the contiguous ranges of dimensions (see Eqs. (1) and (2)). The difference lies in the disappearance of transverse convective terms in the first approximation.

S1.1.2: $\varepsilon \approx \Delta$, i.e., $\frac{1}{\mathrm{Re}_i^{5/4}} < \frac{R_p}{L} \approx \frac{1}{\mathrm{Re}_i}$.

This implies that the respective quantities are developed as power series of Δ in the form of:

$$u = u_0 + \Delta^{1/2} u_1 + \Delta u_2$$
$$v = v_0 + \Delta^{1/2} v_1 + \Delta v_2$$
$$p = p_0 + \Delta^{1/2} p_1 + \Delta p_2 \; ; \qquad (20)$$

S1.1.2.1 – initial approximation

$$\frac{\partial p_0}{\partial x} = \frac{\partial^2 u_0}{\partial y^2}, \quad \frac{\partial p_0}{\partial y} = \frac{\partial^2 v_0}{\partial y^2}, \quad \frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}, = 0 \; ; \qquad (21)$$

S1.1.2.2 – first approximation

$$(\bar{u} + u')\frac{\partial u_0}{\partial x} = -\frac{\partial p_0}{\partial x} - \frac{\partial p_1}{\partial x} + \frac{\partial^2 u_1}{\partial y^2} \qquad (22a)$$

$$(\bar{u} + u')\frac{\partial v_0}{\partial x} = -\frac{\partial p_0}{\partial y} - \frac{\partial p_1}{\partial y} + \frac{\partial^2 v_1}{\partial y^2} \qquad (22b)$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0 \; ; \qquad (22c)$$

S1.1.2.3 – second approximation

$$(\bar{u} + u')\frac{\partial u_1}{\partial x} + v'\frac{\partial u_0}{\partial y} = -\frac{\partial p_0}{\partial x} - \frac{\partial p_1}{\partial x} - \frac{\partial p_2}{\partial x} + \frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2}$$

$$(\bar{u} + u')\frac{\partial v_1}{\partial x} + v'\frac{\partial v_0}{\partial y} = -\frac{\partial p_0}{\partial y} - \frac{\partial p_1}{\partial y} - \frac{\partial p_2}{\partial y} + \frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^2 v_2}{\partial y^2}$$

$$\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} = 0 \, . \qquad (23)$$

This case differs from S1.1.1 in the second approximation. Besides the new longitudinal inertial term, here also viscous longitudinal terms are observed:

S1.1.3: $\varepsilon > \Delta$, i.e., $\frac{1}{\mathrm{Re}_i} < \frac{R_p}{L} < \frac{1}{\mathrm{Re}_i^{7/8}}$.

The respective quantities are expressed as power series in ε and Δ:

$$u = u_0 + (\varepsilon^2/\Delta^{3/2}) u_1 + (\varepsilon^2/\Delta) u_2$$
$$v = v_0 + (\varepsilon^2/\Delta^{3/2}) v_1 + (\varepsilon^2/\Delta) v_2$$
$$p = p_0 + (\varepsilon^2/\Delta^{3/2}) p_1 + (\varepsilon^2/\Delta) p_2 \, . \qquad (24)$$

From (24) and (12) there follows a system of asymptotic equations:

S1.1.3.1 – initial approximation

$$\frac{\partial p_0}{\partial x} = \frac{\partial^2 u_0}{\partial y^2}, \quad \frac{\partial p_0}{\partial y} = \frac{\partial^2 v_0}{\partial y^2}, \quad \frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0 \; ; (25)$$

S1.1.3.2 – first approximation

$$(\bar{u} + u')\frac{\partial u_0}{\partial x} = -\frac{\partial p_0}{\partial x} - \frac{\partial p_1}{\partial x} + \frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2}$$

$$(\bar{u} + u')\frac{\partial v_0}{\partial x} = -\frac{\partial p_0}{\partial y} - \frac{\partial p_1}{\partial y} + \frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2}$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0 \; ; \qquad (26)$$

S1.1.3.3 – second approximation

$$\frac{\partial p_2}{\partial x} = -\frac{\partial p_0}{\partial x} + \frac{\partial^2 u_2}{\partial y^2},$$

$$\frac{\partial p_2}{\partial y} = -\frac{\partial p_0}{\partial y} + \frac{\partial^2 v_2}{\partial y^2}, \quad \frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} = 0 \, . \qquad (27)$$

The viscous character of the initial-approximation equations is still preserved. The role of the longitudinal viscous effects however, is stronger – they appear in the first approximation. Unlike the previous cases, the viscous transverse interaction has a leading role in the second approximation. These facts are connected with the serious disturbance which larger particles initiate in the external flow and, as a result, the transverse interaction becomes stronger. The longitudinal interaction appears only in the first order of approximation as a balance between the convective and viscous terms.

S.1.2.: $\mathrm{Re}_p \approx \Delta$, i.e., $\frac{R_p}{L} \approx \frac{1}{\mathrm{Re}_i^{7/8}}$.

This implies that the respective quantities are developed only as power series of Δ in the form of:

$$u = u_0 + \Delta^{1/2} u_1 + \Delta^{3/4} u_2$$

$$v = v_0 + \Delta^{1/2} v_1 + \Delta^{3/4} v_2$$

$$p = p_0 + \Delta^{1/2} p_1 + \Delta^{3/4} p_2 \; ; \tag{28}$$

S1.2.1 – zero approximation

$$(\bar{u} + u')\frac{\partial u_0}{\partial x} = -\frac{\partial p_0}{\partial x} + \frac{\partial^2 u_0}{\partial y^2}$$

$$(\bar{u} + u')\frac{\partial v_0}{\partial x} = -\frac{\partial p_0}{\partial y} + \frac{\partial^2 v_0}{\partial y^2}$$

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0 \; ; \tag{29}$$

S1.2.2 – first approximation

$$(\bar{u} + u')\frac{\partial u_1}{\partial x} = -\frac{\partial p_0}{\partial x} - \frac{\partial p_1}{\partial x} + \frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2}$$

$$(\bar{u} + u')\frac{\partial v_1}{\partial x} = -\frac{\partial p_0}{\partial y} - \frac{\partial p_1}{\partial y} + \frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2}$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0 \; ; \tag{30}$$

S1.2.3 – second approximation

$$(\bar{u} + u')\frac{\partial u_2}{\partial x} + v'\frac{\partial u_0}{\partial y} + v_0\frac{\partial u'}{\partial y} = -\frac{\partial p_0}{\partial x} - \frac{\partial p_2}{\partial x} + \frac{\partial^2 u_2}{\partial y^2}$$

$$(\bar{u} + u')\frac{\partial v_2}{\partial x} + v'\frac{\partial v_0}{\partial y} = -\frac{\partial p_0}{\partial y} - \frac{\partial p_2}{\partial y} + \frac{\partial^2 v_2}{\partial y^2}$$

$$\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} = 0 \; . \tag{31}$$

For the first time in the governing asymptotic equations (zero approximation) convective terms appear. They balance the transverse viscous effects. The first approximation is the same as in the previous case (S1.1.3.2). The transverse inertial effects are observed just in the second approximation. So, larger particles interact in a more complex manner with the boundary-layer flow. This is a result both of the complicated external flow and the more intensive disturbance that is caused by these larger species.

Thus, the relative dimension of the sphere $R_p/L \approx 1/\mathrm{Re}_i^{7/8}$ is a borderline where the type of interaction transforms from prevailing viscous ($R_p/L < 1/\mathrm{Re}_i^{7/8}$) to mainly inertial ($R_p/L > 1/\mathrm{Re}_i^{7/8}$):

S1.3.: $\mathrm{Re}_p > \Delta$, i.e., $\dfrac{1}{\mathrm{Re}_i^{7/8}} < \dfrac{R_p}{L} < \dfrac{1}{\mathrm{Re}_i^{5/8}}$,

where three additional subcases may be distinguished:

S1.3.1: $\varepsilon^2 < \Delta$, i.e., $\dfrac{1}{\mathrm{Re}_i^{7/8}} < \dfrac{R_p}{L} < \dfrac{1}{\mathrm{Re}_i^{3/4}}$.

The respective presentations of u, v and p are:

$$u = u_0 + (\Delta^{3/2}/\varepsilon^2) u_1 + \Delta^{1/2} u_2$$

$$v = v_0 + (\Delta^{3/2}/\varepsilon^2) v_1 + \Delta^{1/2} v_2$$

$$p = p_0 + (\Delta^{3/2}/\varepsilon^2) p_1 + \Delta^{1/2} p_2 \; . \tag{32}$$

S1.3.1.1 – initial approximation

$$(\bar{u} + u')\frac{\partial u_0}{\partial x} = -\frac{\partial p_0}{\partial x},$$

$$(\bar{u} + u')\frac{\partial v_0}{\partial x} = -\frac{\partial p_0}{\partial y}, \quad \frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0 \; ; \tag{33}$$

S1.3.1.2 – first approximation

$$(\bar{u} + u')\frac{\partial u_1}{\partial x} = -\frac{\partial p_0}{\partial x} - \frac{\partial p_1}{\partial x} + \frac{\partial^2 u_0}{\partial y^2}$$

$$(\bar{u} + u')\frac{\partial v_1}{\partial x} = -\frac{\partial p_0}{\partial y} - \frac{\partial p_1}{\partial y} + \frac{\partial^2 v_0}{\partial y^2}$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0 \; ; \tag{34}$$

S1.3.1.3 – second approximation

$$(\bar{u} + u')\frac{\partial u_2}{\partial x} = -\frac{\partial p_0}{\partial x} - \frac{\partial p_2}{\partial x} + \frac{\partial^2 u_0}{\partial x^2}$$

$$(\bar{u} + u')\frac{\partial v_2}{\partial x} = -\frac{\partial p_0}{\partial y} - \frac{\partial p_2}{\partial y} + \frac{\partial^2 v_0}{\partial x^2}$$

$$\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} = 0 \; . \tag{35}$$

The major effect is the inertial interaction with the boundary layer. The leading system of asymptotic equations (zero approximation) does not include any viscous term. The transverse viscous effect is observed in the first approximation while the longitudinal one appears only in the second approximation. Transverse convectional terms do not appear at all in the investigated order of approximation.

S1.3.2: $\varepsilon^2 \approx \Delta$, i.e., $\frac{R_p}{L} \approx \frac{1}{\mathrm{Re}_i^{3/4}}$

The respective quantities are presented only as power series of Δ:

$$u = u_0 + \Delta^{1/2} u_1 + \Delta u_2$$
$$v = v_0 + \Delta^{1/2} v_1 + \Delta v_2$$
$$p = p_0 + \Delta^{1/2} p_1 + \Delta p_2 \,. \tag{36}$$

Equation (36) and (12) lead to the following sequential approximations:

S1.3.2.1 – initial approximation

$$(\bar{u} + u')\frac{\partial u_0}{\partial x} = -\frac{\partial p_0}{\partial x},$$
$$(\bar{u} + u')\frac{\partial v_0}{\partial x} = -\frac{\partial p_0}{\partial y}, \quad \frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0\,; \tag{37}$$

S1.3.2.2 – first approximation

$$(\bar{u} + u')\frac{\partial u_1}{\partial x} + v_0\frac{\partial u'}{\partial y} = -\frac{\partial p_0}{\partial x} - \frac{\partial p_1}{\partial x} + \frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2}$$
$$(\bar{u} + u')\frac{\partial v_1}{\partial x} = -\frac{\partial p_0}{\partial y} - \frac{\partial p_1}{\partial y} + \frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^2 v_0}{\partial y^2}$$
$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0\,; \tag{38}$$

S1.3.2.3 – second approximation

$$(\bar{u} + u')\frac{\partial u_2}{\partial x} + v'\frac{\partial u_0}{\partial y} + v_1\frac{\partial u'}{\partial y} + v_0\frac{\partial u_0}{\partial y} + u_0\frac{\partial u_0}{\partial x} = -\frac{\partial p_0}{\partial x} - \frac{\partial p_1}{\partial x} - \frac{\partial p_2}{\partial x} + \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2}$$
$$(\bar{u} + u')\frac{\partial v_2}{\partial x} + v'\frac{\partial v_0}{\partial y} + v_0\frac{\partial v_0}{\partial y} + u_0\frac{\partial v_0}{\partial x} = -\frac{\partial p_0}{\partial y} - \frac{\partial p_1}{\partial y} - \frac{\partial p_2}{\partial y} + \frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2}$$
$$\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} = 0\,. \tag{39}$$

The leading-order asymptotic equations contain only longitudinal convectional terms. The transverse viscous and longitudinal inertial changes are observed in the first approximation:

S1.3.3: $\varepsilon^2 > \Delta$, i.e., $\frac{1}{\mathrm{Re}_i^{3/4}} < \frac{R_p}{L} < \frac{1}{\mathrm{Re}_i^{5/8}}$.

The respective power series have the form of:

$$u = u_0 + \varepsilon u_1 + \Delta^{1/2} u_2$$
$$v = v_0 + \varepsilon v_1 + \Delta^{1/2} v_2$$
$$p = p_0 + \varepsilon p_1 + \Delta^{1/2} p_2\,, \tag{40}$$

and therefore the sequential approximations have the form of:

S1.3.3.1 – initial approximation

$$(\bar{u} + u')\frac{\partial u_0}{\partial x} = -\frac{\partial p_0}{\partial x},$$
$$(\bar{u} + u')\frac{\partial v_0}{\partial x} = -\frac{\partial p_0}{\partial y}, \quad \frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0\,; \tag{41}$$

S1.3.3.2 – first approximation

$$(\bar{u} + u')\frac{\partial u_1}{\partial x} + v_0\frac{\partial u'}{\partial y} = -\frac{\partial p_0}{\partial x} - \frac{\partial p_1}{\partial x}$$
$$(\bar{u} + u')\frac{\partial v_1}{\partial x} = -\frac{\partial p_1}{\partial y}$$
$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0\,; \tag{42}$$

S1.3.3.3 – second approximation

$$(\bar{u} + u')\frac{\partial u_2}{\partial x} = -\frac{\partial p_0}{\partial x} - \frac{\partial p_2}{\partial x} + \frac{\partial^2 u_0}{\partial x^2}$$

$$(\bar{u}+u')\frac{\partial v_2}{\partial x} = -\frac{\partial p_0}{\partial y} - \frac{\partial p_2}{\partial y} + \frac{\partial^2 v_0}{\partial x^2}$$

$$\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} = 0 . \qquad (43)$$

The presence of convective terms is again the only effect in the leading-order equations. Just in the second approximation there are viscous terms in the longitudinal direction. Transverse viscous and inertial effects do not appear in the regarded order of approximation.

Therefore, within the assumption of small local Reynolds number ($\mathrm{Re_p} < 1$), three basic types of disturbance field are outlined:

1) Viscous disturbance ($\mathrm{Re_p} < \Delta$; Eqs. (17), (21), and (25)). After simple transformations, these systems of equations are brought into Poisson-type differential equations as in I.

2) Intermediate case ($\mathrm{Re_p} \approx \Delta$; Eq. (29)) where longitudinal inertia arises for the first time. The governing equations (29) have the form of reduced boundary-layer case.

3) Convective disturbance ($\Delta < \mathrm{Re_p} < 1$; Eqs. (33), (37) and (41)) – here the longitudinal convection terms totally dominate the leading order of asymptotic equations and they have the form of reduced potential flow [6].

4. Higher local Reynolds numbers – $\mathrm{Re_p} \approx 1$ and $\mathrm{Re_p} > 1$.

The first possibility is:

$\boxed{\mathrm{S2.}}$: $\mathrm{Re_p} \approx 1, \quad \frac{R_p}{L} \approx \frac{1}{\mathrm{Re}_i^{5/8}}$.

The power series for the respective quantities are:

$$u = u_0 + \Delta^{1/4}u_1 + \Delta^{1/2}u_2$$
$$v = v_0 + \Delta^{1/4}v_1 + \Delta^{1/2}v_2$$
$$p = p_0 + \Delta^{1/4}p_1 + \Delta^{1/2}p_2 ; \qquad (44)$$

S2.1 – initial approximation

$$(\bar{u}+u')\frac{\partial u_0}{\partial x} = -\frac{\partial p_0}{\partial x},$$

$$(\bar{u}+u')\frac{\partial v_0}{\partial x} = -\frac{\partial p_0}{\partial y}, \quad \frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0 ; \qquad (45)$$

S2.2 – first approximation

$$(\bar{u}+u')\frac{\partial u_1}{\partial x} = -\frac{\partial p_0}{\partial x} - \frac{\partial p_1}{\partial x},$$

$$(\bar{u}+u')\frac{\partial v_1}{\partial x} = -\frac{\partial p_1}{\partial y}, \quad \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0 ; \qquad (46)$$

S2.3 – second approximation

$$(\bar{u}+u')\frac{\partial u_2}{\partial x} + v_0\frac{\partial u_0}{\partial y} + u_0\frac{\partial u_0}{\partial x} = -\frac{\partial p_0}{\partial x} - \frac{\partial p_1}{\partial x} - \frac{\partial p_2}{\partial x} + \frac{\partial^2 u_0}{\partial x^2}$$

$$(\bar{u}+u')\frac{\partial v_2}{\partial x} + v_0\frac{\partial v_0}{\partial y} + u_0\frac{\partial v_0}{\partial x} = -\frac{\partial p_0}{\partial y} - \frac{\partial p_2}{\partial y} + \frac{\partial^2 v_0}{\partial x^2}$$

$$\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} = 0 . \qquad (47)$$

Here, convective terms totally dominate the leading order of equations and the first order correction. Longitudinal viscous effects occur only in the second approximation.

$\boxed{\mathrm{S3.}}$: $\mathrm{Re_p} > 1, \quad \text{i.e.,} \quad \frac{1}{\mathrm{Re}_i^{5/8}} < \frac{R_p}{L} < \frac{1}{\mathrm{Re}_i^{1/2}}$.

The solution is sought in power series of ε:

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2$$
$$v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2$$
$$p = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 ; \qquad (48)$$

S3.1 – initial approximation

$$(\bar{u}+u')\frac{\partial u_0}{\partial x} = -\frac{\partial p_0}{\partial x},$$

$$(\bar{u}+u')\frac{\partial v_0}{\partial x} = -\frac{\partial p_0}{\partial y}, \quad \frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0 ; \qquad (49)$$

S3.2 – first approximation

$$(\bar{u}+u')\frac{\partial u_1}{\partial x} + v_0\frac{\partial u'}{\partial y} = -\frac{\partial p_0}{\partial x} - \frac{\partial p_1}{\partial x}$$

$$(\bar{u}+u')\frac{\partial v_1}{\partial x} = -\frac{\partial p_0}{\partial y} - \frac{\partial p_1}{\partial y}$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0 \; ; \qquad (50)$$

S3.3 – second approximation

$$(\bar{u}+u')\frac{\partial u_2}{\partial x} + v_0\frac{\partial u_0}{\partial y} + u_0\frac{\partial u_0}{\partial x} + v_1\frac{\partial u'}{\partial y} = -\frac{\partial p_0}{\partial x} - \frac{\partial p_1}{\partial x} - \frac{\partial p_2}{\partial x}$$

$$(\bar{u}+u')\frac{\partial v_2}{\partial x} + v_0\frac{\partial v_0}{\partial y} + u_0\frac{\partial v_0}{\partial x} = -\frac{\partial p_0}{\partial y} - \frac{\partial p_2}{\partial y}$$

$$\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} = 0 \; . \qquad (51)$$

Here, only convective terms are observed. The longitudinal inertial effects are the basic result of the interaction. This is the first range of dimensions, where the asymptotic equations do not possess any viscous terms to the second order approximation.

Conclusions

The asymptotic model for the interaction of a larger neutrally buoyant particle (Eqs. (12)) with the boundary-layer flow past a plate exhibits the following characteristic features:

1) With neutral buoyancy the only possible scaling of the longitudinal disturbance velocity is the viscous deformation of Faxen's type, i.e., $u_f > u_s$. The characteristic change of the transverse disturbance velocity is scaled with a quantity of Saffman's type, i.e., $v_s > v_f$.

2) The basic effect of the perturbation of the external flow field, due to the presence of the suspended particle, depends on its dimensions. For smaller particles (S1.1), the leading terms in the asymptotic equations have viscous transversal character. For larger particles however (S1.3, S2, S3), the leading effect is of longitudinal inertial nature.

3) In the sequential approximations a further differentiation of the interaction type for the particles according to their dimensions is observed. For smaller particles, more important are the inertial (longitudinal and transverse) effects, followed by the longitudinal viscous interactions. For larger particles, however, the opposite is observed – the role of the inertial effects is further increased.

Generally speaking, for larger particles the role of the viscous transversal migration is continuously decreasing and the longitudinal inertial interaction begins to determine the behavior of the particles in the boundary-layer flow past a plate.

Acknowledgements

The Bulgarian National Fund of Scientific Research (Project X-244) is gratefully acknowledged for financial support.

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Received June 29, 1993;
accepted March 2, 1994

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